Products of Prime Powers in Binary Recurrence Sequences
Part II: The Elliptic Case, with an Application to a Mixed Quadratic-Exponential Equation

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Abstract. In Part I the diophantine equation $G_n = wp_1^{m_1} \cdots p_l^{m_l}$ was studied, where $\{G_n\}_{n=0}^{\infty}$ is a linear binary recurrence sequence with positive discriminant. In this second part we extend this to negative discriminants. We use the $p$-adic and complex Gelfond-Baker theory to find explicit upper bounds for the solutions of the equation. We give algorithms to reduce those bounds, based on diophantine approximation techniques. Thus we have a method to solve the equation completely for arbitrary values of the parameters. We give an application to a quadratic-exponential equation.


6A. Introduction. It is assumed that the reader is familiar with Part I of this paper (Pethò and de Weger [4]). We adopt notations and assumptions from Part I without further reference.

In Part I we studied Eq. (1.1):

$$G_n = wp_1^{m_1} \cdots p_l^{m_l},$$

for $\Delta > 0$. The $p$-adic Gelfond-Baker theory, together with a trivial observation on the exponential growth of $|G_n|$, provided us with upper bounds for the solutions. In the case $\Delta < 0$, which is our present topic, the situation is essentially more complicated. The $p$-adic behavior of $G_n$ does not depend on the sign of the discriminant. But in the case $\Delta < 0$, the growth of $|G_n|$ is not as nice as in the case $\Delta > 0$. However, information on its growth can be obtained from the complex Gelfond-Baker theory. The fact that Eq. (1.1) has only finitely many solutions was shown by Mahler [3].

Section 7 is devoted to the complex arguments. In it we solve the diophantine inequality $|G_n| \leq v$ for a fixed $v$. An upper bound for $n$ is given that has particularly good dependence on $v$. We present algorithms to reduce this upper bound, so that the inequality can be solved completely in any practical case. These algorithms are not new; they come essentially from Baker and Davenport [1] and Cijsouw, Korlaar, and Tijdeman (appendix to Stroeker and Tijdeman [5]).
In Subsection 8A we combine the results of Sections 3 and 7 to obtain explicit upper bounds for (1.1). In Subsection 8B an algorithm is presented to reduce these upper bounds. It is a combination of the algorithms of Sections 4 and 7. We give an example in Subsection 8C. Finally, in Section 9 we present an application to a certain type of mixed quadratic-exponential diophantine equation.

6B. Preliminaries. Let in the sequel $\Delta < 0$. Since $\alpha/\beta$ is not a root of unity, $B > 2$. Since $(\alpha, \beta)$ and $(\lambda, \mu)$ are pairs of complex conjugates, $|\alpha| = |\beta|$ and $|\lambda| = |\mu|$. Thus $L = \log \max(|eD|^{1/4}, |a\lambda\sqrt{D}|)$. Lemmas 3.2, 4.2, and 4.3 hold also for $\Delta < 0$.

As in the case $\Delta > 0$, we have to exclude the case where only finitely many $p_r$-adic digits of $\theta$, are nonzero. Let $\rho = \frac{1}{2}(1 + \sqrt{-3})$.

**Lemma 6.1.** If only finitely many $p_r$-adic digits $u_{i,r}$ of $\theta_i$ are nonzero, then $\theta_i = 0$, and $G_n = \pm R_n, \kappa S_n, \kappa T_n$ or $\kappa U_n$, where $\kappa \in \mathbb{Q}$, and

$$R_n = (\alpha^n - \beta^n)/(\alpha - \beta), \quad S_n = \alpha^n + \beta^n,$$

$$T_n = (1 \pm \sqrt{-1})\alpha^n + (1 \mp \sqrt{-1})\beta^n,$$

$$U_n = (1 \pm \omega)\alpha^n + (1 \pm \bar{\omega})\beta^n, \quad \omega = \rho + \bar{\rho}.$$  

The case $G_n = \kappa T_n$ can occur only if $d = -1$, and $G_n = \kappa U_n$ only if $d = -3$.

**Proof.** As in the proof of Lemma 4.4, $\theta_i = r \in \mathbb{Z}$, and $(\beta/\alpha)'(\mu/\lambda) = \eta$ is a root of unity. Then $\eta\lambda\alpha' = \mu\beta'$, hence

$$G_n = \lambda\alpha'(\alpha^{n-r} + \eta\beta^{n-r}).$$

Recall that $B = \alpha\beta > 2$. Notice that

$$G_0 B(\eta\alpha'^{-1} + \beta'^{-1}) = G_1(\eta\alpha' + \beta').$$

By $(B, G_1) = 1$, it follows that $\alpha\beta|\eta\alpha' + \beta'$. By $(A, B) = 1$, we have $(\alpha, \beta) = (1)$, and from $\alpha|\beta'$ it then follows that $\theta = r = 0$. So $G_0 = \lambda(1 + \eta) \in \mathbb{Z}$. Then $\lambda = \kappa(1 + \bar{\eta})$ for some $\kappa \in \mathbb{Q}$. Choose $\kappa$ such that $G_0, G_1 \in \mathbb{Z}$ and $(G_0, G_1) = 1$. Now the result follows easily, since for $\eta$ there are only the possibilities $\pm 1$, and $\pm \sqrt{-1}$ if $d = -1$, and $\pm \rho, \pm \bar{\rho}$ if $d = -3$. $\square$

In the cases of Lemma 6.1, Eq. (1.1) can be treated as follows. The smallest index $n = g(mp^l)$ such that $mp^l|G_n$ grows exponentially with $l$. Also $G_n$ grows exponentially with $n$ (see Theorem 7.2). Hence $G_{g(mp^l)}$ grows double exponentially with $l$. It follows that $wp_1^{m_1} \cdots p_t^{m_t}$ cannot keep up with $G_{g(wp_1^{m_1} \cdots p_t^{m_t})}$. So, if $m_1, \ldots, m_t$ are large enough, there is a prime $q$ such that $q|G_{g(wp_1^{m_1} \cdots p_t^{m_t})}$, but $q \nmid wp_1^{m_1} \cdots p_t^{m_t}$. Now the special properties of the sequences $R_n$, $S_n$, $T_n$, and $U_n$ can be employed to prove that $q|G_n$ whenever $wp_1^{m_1} \cdots p_t^{m_t}|G_n$. We illustrate this with an example.

Let $A = 5$, $B = 13$, $G_0 = G_1 = 1$. Then $\Delta = -27$, $\alpha = 1 + 3\rho$, $\lambda = (1 + \rho)/3$. We solve $G_n = \pm 2^n$. The sequence $G_n = \lambda\alpha^n + \bar{\lambda}\bar{\alpha}^n$ is related to the sequence $H_n = \lambda\alpha^n + \bar{\alpha}^n$. In fact, we have $G_n H_n R_n = R_{3n}/3$. Since $R_n$ has nice divisibility properties, we thus have information on the prime divisors of $G_n$ and $H_n$. We find:

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_n$</td>
<td>1</td>
<td>1</td>
<td>-8</td>
<td>-53</td>
<td>-161</td>
<td>-116</td>
<td>1513</td>
<td>9073</td>
<td>25696</td>
</tr>
<tr>
<td>$H_n$</td>
<td>1</td>
<td>4</td>
<td>7</td>
<td>-17</td>
<td>-176</td>
<td>-659</td>
<td>-1007</td>
<td>3532</td>
<td>30751</td>
</tr>
<tr>
<td>$R_n$</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>12</td>
<td>-5</td>
<td>-181</td>
<td>-840</td>
<td>-1847</td>
<td>1685</td>
</tr>
</tbody>
</table>
Now \( G_n = 0 \mod 16 \) if and only if \( n \equiv 8 \mod 12 \), \( H_n = 0 \mod 16 \) if and only if \( n \equiv 4 \mod 12 \), and \( R_n \equiv 0 \mod 16 \) if and only if \( n \equiv 0 \mod 12 \). Further, \( G_n H_n R_n = R_{12} \mod 3 = -2^4 \cdot 7 \cdot 11 \cdot 23 \), and it follows that \( 2^4 \cdot 7 \cdot 11 \cdot 23 \mid G_n H_n \) for all \( n \equiv 0 \mod 4 \). In fact, \( 11 \mid G_n \) whenever \( 16 \mid G_n \). Thus \( G_n = \pm 2^m \) implies \( m \leq 3 \). In the next section we show how to solve \( |G_n| \leq 8 \).

Another way to treat (1.1) in the case \( \theta_i = 0 \) is the following. By Lemma 4.2, \( m \leq g_i + 1 + \text{ord}_p(n) \). Hence,

\[ |G_n| = |w| p_1^{m_1} \cdots p_{r}^{m_r} \leq v_0 n \]

for some constant \( v_0 \). Only minor changes in the arguments of Section 7 suffice to deal with this inequality, instead of \( |G_n| \leq v \).

### 7. The Growth of the Recurrence Sequence.

#### 7A. Application of a Theorem of Waldschmidt

In this subsection we study the diophantine inequality

\[ |G_n| \leq v \]

for a fixed \( v \in \mathbb{R}, v \geq 1 \). We apply a result of Waldschmidt [6] from the complex Gelfond-Baker theory, which gives an upper bound for \( n \) that is particularly good in \( v \). See also Kiss [2].

Let \( a_0 \) for \( \xi \in \mathbb{Q}(\sqrt{\Delta}) \) be the leading coefficient of its minimal polynomial. We define the height of \( \xi \) by

\[ h(\xi) = \frac{1}{2} \log a_0 + \log \max(1, |\xi|) \]

in accordance with Waldschmidt's height function (cf. [6, p. 259]). Let \( \alpha_1, \ldots, \alpha_n \in \mathbb{Q}(\sqrt{\Delta}), b_1, \ldots, b_n \in \mathbb{Z} \). Put

\[ \Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n, \]

where \( \log \) denotes the principal value of the complex logarithm, i.e., \( -\pi < \text{Im} \log z \leq \pi \). Assume \( \Lambda \neq 0 \). Let \( V_1, \ldots, V_n \) be real numbers with \( \frac{1}{2} \leq V_1 \leq \cdots \leq V_n \), and \( V_i \geq \max\{h(\alpha_i), \frac{1}{2} |\log \alpha_i|\} (i = 1, \ldots, n) \). Put \( W = \max_{1 \leq i \leq n} \log|b_i| \). Define \( V_i^+ = \max(1, V_i) \) for \( i = n - 1, n \). Put

\[ C_4 = 2^{9n+53n^2}V_1 \cdots V_n \log(2eV_n^+), \quad C_5 = C_4 \log(2eV_n^+). \]

**Theorem 7.1 (Waldschmidt).** With the above definitions,

\[ \left| \Lambda \right| > \exp\{-(C_4W + C_5)\}. \]

We apply this to (7.1) as follows. Let

\[ E = -\lambda \mu \Delta, \]

\[ U_2 = \frac{1}{2} \max(\pi, \log B), \quad U_3 = \frac{1}{2} \max(\pi, \log E), \]

\[ U_2^+ = \min(U_2, U_3), \quad U_3^+ = \max(U_2, U_3), \]

\[ C_4' = 2^{9n^2}U_2U_3 \log(2eU_2^+), \quad C_5' = C_4 \log(4eU_3^+), \]

\[ C_6 = (\log(\pi/2|\mu|)) + C_5' + C_4' \log(4C_4'/\log B) \times 4/\log B. \]
THEOREM 7.2. Let \( v \in \mathbb{R}, \ v \geq 1 \). Then all solutions \( n > 0 \) of (7.1) satisfy
\[
n < C_6 + \frac{4}{\log B} \log \max(v, 2|G_0 \mu \sqrt{\Delta}|).
\]

Remark. Notice that \( C_6 \) does not depend on \( v \).

Proof. By \( \Delta < 0 \), both \((\alpha, \beta)\) and \((\lambda, \mu)\) are pairs of complex conjugates. Hence
\[
|\alpha| = |\beta| = B^{1/2} \geq \sqrt{2}.
\]
We have from (7.1)
\[
(7.2) \quad \left| \left( \frac{-\lambda}{\mu} \right)^n \left( \frac{\alpha}{\beta} \right) - 1 \right| \leq \frac{v}{|\mu|} B^{-n/2}.
\]
We may assume \( n \geq 2 \). Let \( -\lambda/\mu = e^{2\pi i \phi}, \alpha/\beta = e^{2\pi i \phi} \), with \( -\frac{1}{2} < \phi \leq \frac{1}{2}, \ -\frac{1}{2} < \psi \leq \frac{1}{2} \). Let \( k_0, k_1 \in \mathbb{Z} \) be such that \( |j \psi + n \phi + k_j| \leq \frac{1}{2} \). Then \( |k_j| \leq 1 + \frac{1}{2} n \leq n \) (\( j = 0, 1 \)). Put
\[
\Lambda_j = 2\pi i (j \psi + n \phi + k_j) = j \log \left( \frac{-\lambda}{\mu} \right) + n \log \left( \frac{\alpha}{\beta} \right) + 2k_j \log(-1)
\]
for \( j = 0, 1 \). It is an easy exercise to show that \( |x| \leq \frac{1}{2} |e^{2\pi i x} - 1| \) holds for all \( x \in \mathbb{R} \) with \( |x| \leq \frac{1}{2} \). Now, from (7.2) we have an upper bound for \( |\Lambda_1| \):
\[
|\Lambda_1| = 2\pi |\psi + n \phi + k_1| \leq \frac{\pi}{2} |e^{2\pi i (\psi + n \phi + k_1)} - 1|
\]
\[= \frac{\pi}{2} \left| \frac{-\lambda}{\mu} \right|^n |\frac{\alpha}{\beta} - 1| \leq \frac{\pi}{2|\mu|} v B^{-n/2}.
\]
It may happen that \( \Lambda_1 = 0 \). In that case, \( \psi + n \phi \in \mathbb{Z} \), hence \(-\lambda/\mu)(\alpha/\beta)^n = 1\), and it follows that \( G_n = \lambda \alpha^n + \mu \beta^n = 0 \). Kiss [2] showed that this implies \( |R_n| \leq 2|G_0| \), where \( R_n = (\alpha^n - \beta^n)/(\alpha - \beta) \). From this, Kiss derived an upper bound for \( n \). We shall follow his argument, but we apply another, sharper result from the Gelfond-Baker theory than Kiss. Notice that, by \( |\beta| = B^{1/2} \),
\[
2|G_0| \geq |R_n| = \frac{B^{n/2} |(\lambda)^n - 1|}{|\Delta|} \geq \frac{4B^{n/2}}{|\Delta|} |\phi n + k_0| = \frac{2B^{n/2}}{\pi \sqrt{|\Delta|}} |\Lambda_0|.
\]
Now \( \Lambda_0 \neq 0 \), since by \( n \geq 2 \) the contrary would imply \( \phi \in \mathbb{Q} \), which is impossible, since \( \alpha/\beta \) is not a root of unity. Thus, take \( j = 1 \) if \( \Lambda_1 \neq 0 \), and \( j = 0 \) otherwise. Then \( \Lambda_j \neq 0 \), and
\[
|\Lambda_j| \leq \frac{\pi}{2|\mu|} \max(v, 2|G_0 \mu \sqrt{\Delta}|) B^{-n/2}.
\]
From Theorem 7.1 we can derive a lower bound for \( |\Lambda_j| \). Notice that \( \max(j, n, 2|k_j|) \leq 2n \), so that \( W = \log(2n) \). We choose \( V_1 = \frac{1}{2} \). The number \( \alpha/\beta \) satisfies
\[
Bx^2 - (A^2 - 2B)x + B = 0,
\]
hence \( h(\alpha/\beta) \leq \frac{1}{2} \log B \). And \(-\lambda/\mu \) satisfies
\[
Ex^2 - (2E + \Delta G_0^2)x + E = 0,
\]
hence \( h(-\lambda/\mu) \leq \frac{1}{2} \log E \). Thus \( V_2 = U_2^+, \ V_3 = U_3^+ \) satisfy the requirements for Theorem 7.1. We find
\[
|\Lambda_j| \geq \exp\left\{-C_3'(\log(2n) + \log(2eU_3^+))\right\}
\]
\[= \exp\left\{-\left(C_4' \log n + C_5\right)\right\}.
\]
Combining (7.3) and (7.4) we find \( n < a + b \log n \), where

\[
a = \frac{2}{\log B} \left( \log \max \left( v, 2|G_0 \mu | \right) + \log \frac{\pi}{2|\mu|} + C_5' \right),
\]

\[
b = 2C_4'/\log B.
\]

The result follows from Lemma 2.2 (Part I), since

\[
b = 2C_4'/\log B = 2^{383} \max \left( \pi, \log B \right) \max \left( \pi, \log E \right) \log(2eU_2^*),
\]

which is certainly larger than \( e^2 \). \(\square\)

We now want to reduce the bound from Theorem 7.2. We do this by studying the diophantine inequality

\[
(7.5) \quad |\psi_j + n\phi + k_j| < v_0 B^{-n/2},
\]

where \( \psi_j = j\psi \) and \( v_0 = \max(v, 2|G_0 \mu |\delta)/4|\mu| \). We have to distinguish between \( \psi_j = 0 \) (the homogeneous case) and \( \psi_j \neq 0 \) (the inhomogeneous case).

7B. The Homogeneous Case. We first study the easier case \( \psi_j = 0 \). We have the following algorithm. Let \( N \) be an upper bound for the solutions of (7.5), for example the bound found in Theorem 7.2.

**Algorithm B** (reduces given upper bound for (7.5) in the case \( \psi_j = 0 \)).

**Input:** \( \phi, B, |\mu|, v_0, N \).

**Output:** new, better bound \( N^* \) for \( n \).

(i) (initialization) Choose \( n_0 \geq 2/\log B \) such that \( B^{n_0/2}/n_0 \geq 2v_0 \); \( N_0 := [N] \); compute the continued fraction

\[
|\phi| = [0, a_1, a_2, \ldots, a_{l_0+1}, \ldots]
\]

and the denominators \( q_1, \ldots, q_{l_0+1} \) of the convergents of \( |\phi| \), with \( l_0 \) so large that \( q_{l_0} \leq N_0 < q_{l_0+1} \); \( i := 0 \);

(ii) (compute new bound) \( A_i := \max(a_1, \ldots, a_{l_i+1}) \); compute the largest integer \( N_{i+1} \) such that

\[
B^{N_{i+1}/2}/N_{i+1} \leq v_0 (A_i + 2);
\]

and \( l_{i+1} \) such that \( q_{l_{i+1}} \leq N_{i+1} < q_{l_{i+1}+1} \);

(iii) (terminate loop)

if \( n_0 \leq N_{i+1} < N \) then \( i := i + 1 \), goto (ii);

else \( N^* := \max(n_0, N_{i+1}) \), stop. \n

**Lemma 7.3.** Algorithm B terminates. Inequality (7.5) with \( \psi_j = 0 \) has no solutions with \( N^* < n < N \).

**Proof.** Termination is trivial, since all \( N_i \) are integers. Notice that \( B^{1/2}/x \) is an increasing function for \( x \geq 2/\log B \). Hence, if \( n \geq n_0 \),

\[
|\phi| - |k_j|/n \leq v_0 B^{-n/2}/n < 1/2n^2.
\]

It follows that \( |k_j|/n \) is a convergent of \( |\phi| \), say \( |k_j|/n = p_m/q_m \). Then \( q_m \leq n \), and, as is well known,

\[
|\phi| - p_m/q_m > 1/(a_{m+1} + 2)q_m^2.
\]
Suppose \( n \leq N_i \) for some \( i \geq 0 \). Then \( m \leq l_i \). Hence,

\[
B^{n/2}/n \leq v_0n^{-2} | \phi | - | k_j |/n |^{-1} < v_0(a_{m+1} + 2) \leq v_0(A_m + 2).
\]

It follows that if \( N_{i+1} \geq n_0 \), then \( n \leq N_{i+1} \). □

We notice that the above algorithm is similar to those of Cijssouw, Korlaar, and Tijdeman (appendix to Stroeker and Tijdeman [5]), and of D. C. Hunt and A. J. van der Poorten (unpublished manuscript).

7C. The Inhomogeneous Case. In the more complicated case \( \psi_j \neq 0 \), we use a technique due to H. Davenport (see Baker and Davenport [1, pp. 133–134]). Again, let \( N \) be an upper bound for \( n \).

**Algorithm C** (reduces upper bound for (7.5) in the case \( \psi_j \neq 0 \)).

**Input:** \( \phi, \psi_j, B, v_0, N \).

**Output:** new, better upper bound \( N^* \) for all but a finite number of explicitly given \( n \).

(i) (initialization) \( N_0 := \lfloor N \rfloor \); compute the continued fraction

\[
| \phi | = [0, a_1, \ldots, a_{l_0}, \ldots]
\]

and the convergents \( p_i/q_i \) \((i = 1, \ldots, l_0)\), with \( l_0 \) so large that \( q_{l_0} > 4N_0 \) and \( \|q_{l_0} \psi_j\| > 2N_0/q_{l_0} \). (If such \( l_0 \) cannot be found within reasonable time, take \( l_0 \) so large that \( q_{l_0} > 4N_0 \); \( i := 0 \);

(ii) (compute new bound)

\[
\text{if } \|q_i \psi_j\| > 2N_i/q_i \text{ then } N_{i+1} := \lfloor 2\log(q_i^2v_0/N_i)/\log B \rfloor;
\]

\[
\text{else compute } K \in \mathbb{Z} \text{ with } |K - q_i \psi_j| \leq \frac{1}{2};
\]

\[
\text{compute } n_0 \in \mathbb{Z}, 0 \leq n_0 < q_i, \text{ with}
\]

\[
K + n_0p_i \equiv 0 \pmod{q_i},
\]

\[
\text{if } n = n_0 \text{ is a solution of } (7.5), \text{ then print an appropriate message;}
\]

\[
N_{i+1} := \lfloor 2\log(4q_i v_0)/\log B \rfloor;
\]

(iii) (terminate loop)

\[
\text{if } N_{i+1} < N_i \text{ then } i := i + 1;
\]

\[
\text{compute the minimal } l_i < l_{i-1} \text{ such that } q_i > 4N_i \text{ and } \|q_i \psi_j\| > 2N_i/q_i. \text{ (If such } l_i \text{ does not exist, choose the minimal } l_i \text{ such that } q_i > 4N_i);
\]

\[
\text{goto (ii)};
\]

\[
\text{else } N^* := N_i, \text{ stop.}
\]

**Lemma 7.4.** Algorithm C terminates. Inequality (7.5) with \( \psi_j \neq 0 \) has for \( N^* < n < N \) only the finitely many solutions found by the algorithm.

**Proof.** It is clear that the algorithm terminates. Suppose that \( n \leq N_i \) for some \( i \geq 0 \). Then if \( \|q_i \psi_j\| > 2N_i/q_i \), we have

\[
\|q_i \psi_j\| = \|q_i(\psi_j + n\phi + k_j) - n\phi q_i\| \leq q_i|\psi_j + n\phi + k_j| + n/q_i \leq q_i v_0 B^{-n/2} + N_i/q_i.
\]
It follows that \( n \leq N_{i+1} \). If \( \|q_i \psi_i\| \leq 2N_i/q_i \), then
\[
|K + np_i + k_j q_i| \leq |K - q_i \psi_j| + q_i |\psi_j + n\phi + k_j| + n|p_i - q_i \phi| \\
\leq \frac{1}{2} + q_i v_0 B^{-n/2} + N_i/q_i < \frac{3}{4} + q_i v_0 B^{-n/2}.
\]
Suppose that \( q_i v_0 B^{-n/2} \leq \frac{1}{4} \). Then \( K + np_i + k_j q_i = 0 \), since it is an integer. By \( (p_i, q_i) = 1 \) it follows that \( n \equiv n_0 \pmod{q_i} \). Since \( q_i \geq N_i \), \( n = n_0 \) is the only possibility. Suppose next that \( q_i v_0 B^{-n/2} > \frac{1}{4} \). Then \( n \leq N_{i+1} \) follows immediately.

We remark that in practice one almost always finds an \( l_i \) such that \( \|q_i \psi_j\| > 2N_i/q_i \), if \( N_i \) is large enough.

8. How to Solve (1.1).

8A. Bounds for the Solutions. Combining the results from the \( p \)-adic and the complex Gelfond-Baker theory (Lemma 3.2 and Theorem 7.2), we now derive upper bounds for the solutions of (1.1) with \( \Delta < 0 \).

**Theorem 8.1.** Put \( C_1 = \max_{1 \leq i \leq t}(C_{1,i}) \) and \( P = p_1 \cdots p_t \). Further, put
\[
C_7 = \max \left\{ C_6 + \frac{4}{\log B} \log(2|G_0 \mu \sqrt{\Delta}|), \right. \\
8 \left. \left( C_6 + \frac{4 \log |w|}{\log B} \right)^{1/3} + \left( \frac{4C_1 \log P}{\log B} \right)^{1/3} \log \left( \frac{108C_1 \log P}{\log B} \right), \right\} \\
C_{b,i} = C_{1,i}(\log C_7)^3 \quad (i = 1, \ldots, t).
\]

Then all solutions of (1.1) satisfy
\[
n < C_7, \quad m_i < C_{b,i} \quad (i = 1, \ldots, t).
\]

**Proof.** From Lemma 3.2 and Theorem 7.2 with \( v = |w| p_1^{m_1} \cdots p_t^{m_t} \), we see that
\[
n < C_6 + \frac{4}{\log B} \log(2|G_0 \mu \sqrt{\Delta}|),
\]
or
\[
n < C_6 + \frac{4 \log |w|}{\log B} + \frac{4C_1 \log P}{\log B} (\log n)^3.
\]

The result now follows from Lemma 2.2 if \( 4C_1 \log P/\log B > (e^2/3)^3 \). This is certainly true.

8B. The Algorithm. We present an algorithm to reduce upper bounds for the solutions of Eq. (1.1). The idea is to apply alternatingly algorithms A and one of B and C. Let \( N \) be an upper bound for \( n \), for example \( N = C_7 \).

**Algorithm D** (reduces upper bounds for the solutions of (1.1)).

**Input:** \( \alpha, \beta, \lambda, \mu, w, p_1, \ldots, p_t, N \).

**Output:** new, better bounds \( N^*, M_i \) for \( n \) and \( m_i \) (\( i = 1, \ldots, t \)).

(i) (initialization) \( N_0 := \lceil N \rceil \); \( j := 1 \);

\[
g_i := \text{ord}_{p_i}(\lambda) + \text{ord}_{p_i}(\log_{p_i}(\alpha/\beta)) \begin{cases} \\
3/2 & \text{if } p_i = 2 \\
1 & \text{if } p_i = 3 \\
1/2 & \text{if } p_i \geq 5 \end{cases} \quad (i = 1, \ldots, t); \\
h_i := \text{ord}_{p_i}(\lambda) + \begin{cases} \\
3/2 & \text{if } p_i = 2 \\
1 & \text{if } p_i = 3 \\
1/2 & \text{if } p_i \geq 5 \end{cases} \quad (i = 1, \ldots, t); \\
\]
(ii) (computation of the $\theta_i$'s, $\phi$ and $\psi$)
compute for $i = 1, \ldots, t$ the first $r_i$ $p_i$-adic digits of
$$\theta_i = -\log_{p_i}((-\lambda/\mu)/\log_{p_i}(\alpha/\beta)) = \sum_{l=0}^{\infty} u_{i,l} p_i^l,$$
where $r_i$ is so large that $p_i^{r_i} \geq N_0$ and $u_{i,r_i} \neq 0$; compute $\psi = \log(-\lambda/\mu)/2\pi i$, and the continued fraction
$$|\phi| = \left|\frac{1}{2\pi i} \log(\alpha/\beta)\right| = \left[0, a_1, \ldots, a_l, \ldots\right]$$
with the convergents $p_i/q_i$ ($i = 1, \ldots, l_0$), where $l_0$ is so large that $q_{l_0-1} \leq N_0 < q_{l_0}$ if $\psi = 0$; $q_{l_0} > 4N_0$ and $||q_{l_0}\psi|| > 2N_0/q_{l_0}$ if $\psi \neq 0$ and such $l_0$ can be found in a reasonable amount of time, $q_{l_0} > 4N_0$ otherwise.

(iii) (one step of Algorithm A)
$M_{i,j} := \max(h_{i,j} + g_{i,j} + \min\{s \in \mathbb{Z}; s \geq 0 \text{ and } p_i^{s} > N_{i,j-1} \text{ and } u_{i,s} \neq 0\})$ ($i = 1, \ldots, t$);

(iv) (one step of Algorithm B or C)
if $\psi = 0$ then $A := \max(a_1, \ldots, a_{l_0-1})$; $v := |w|p_1^{M_{1,1}} \cdots p_t^{M_{t,1}}$;
choose $n_0 \geq 2/\log B$ such that $B^{n_0/2}/n_0 \geq v/2|\mu|$;
compute the largest integer $N_j$ such that $B^{N_j/2}/N_j \leq (A + 2)v/4|\mu|$;
$N_j := \max(n_0, N_j)$;
if $N_j < N_{j-1}$ then compute $l_j$ such that
$q_{l_j-1} \leq N_j < q_{l_j}$;
$j := j + 1$; goto (iii);
else if $||q_{l_j-1}\psi|| > 2N_{j-1}/q_{l_j-1}$ then $N_j := \lceil 2\log(q_{l_j-1}^2v/4|\mu|N_{j-1}^2)/\log B \rceil$;
else compute $K \in \mathbb{Z}$ with $|K - q_{l_j-1}\psi| \leq \frac{1}{2}$;
compute $n_0 \in \mathbb{Z}$, $0 \leq n_0 < q_{l_j-1}$,
with $K + n_0 p_{l_j-1} \equiv 0 \pmod{q_{l_j-1}}$;
if $n = n_0$ is a solution of (1.1) then print an appropriate message;
$N_j := \lceil 2\log(q_{l_j-1}v/|\mu|)/\log B \rceil$;
if $N_j < N_{j-1}$ then compute the minimal $l_j < l_{j-1}$ such that
$q_{l_j} > 4N_j$ and $||q_{l_j}\psi|| > 2N_j/q_{l_j}$ (if such $l_j$ does not exist, choose the minimal $l_j$ such that $q_{l_j} > 4N_j$);
$j := j + 1$; goto (iii);

(v) (termination) $N^* := N_{i,j-1}$; $M_i := M_{i,j}$ ($i = 1, \ldots, t$); stop.

Theorem 8.2. Algorithm D terminates. Equation (1.1) has no solutions with $N^* < n < N$ and $M_i > M_i$ ($i = 1, \ldots, t$), apart from those spotted by the algorithm.

Proof. Clear, from the proofs of Lemmas 7.3 and 7.4.

8C. An Example. Let $A = 1$, $B = 2$, $G_0 = 2$, $G_1 = 3$, then $\Delta = -7$, $\alpha = (1 + \sqrt{-7})/2$, $\lambda = (2 + \sqrt{-7})/\sqrt{-7}$. Let $w = \pm 1$, $p_1 = 3$, $p_2 = 7$. We have with $n_0 = 2$: $C_1 < 6.40 \times 10^{16}$, $C_6 < 9.14 \times 10^{29}$, $C_7 < 7.42 \times 10^{30}$, $C_8 < 2.30 \times 10^{32}$.
Further, \( g_1 = 1, g_2 = 0, h_1 = 1, h_2 = 0 \). Let \( N_0 = 7.42 \times 10^{30} \). We have

\[
\phi = \frac{\log(\alpha/\beta)}{2\pi i} = \left( \pi - \arctan(\sqrt{7}/3) \right)/2\pi
\]

\[
\psi = \frac{\log(-\lambda/\mu)}{2\pi i} = \left( \pi - \arctan(4\sqrt{7}/3) \right)/2\pi
\]

Now, \( M_{11} = 67, M_{21} = 37 \); we choose \( l_0 = 61 \), since

\[
q_{61} = 142 51183 31142 44361 19375 51238 81743 > 4N_0,
\]

and \( \|q_{61}\psi\| = 0.24487 \ldots > 2N_0/q_{61} = 0.104 \ldots \). So we find \( N_1 = 637 \). Next, \( M_{12} = 7, M_{22} = 4 \); we choose \( l_1 = 9 \), since \( q_9 = 10102 > 4 \times 637 \), and \( \|q_9\psi\| = 0.38745 \ldots > 2 \times 637/10102 \). So we find \( N_2 = 74 \). Next, \( M_{13} = 6, M_{23} = 3 \); we choose \( l_2 = 6 \), since \( q_6 = 1291 > 4 \times 74 \), and \( \|q_6\psi\| = 0.49398 \ldots > 2 \times 74/1291 \). So we find \( N_3 = 60 \). In the next step we find no improvement. Hence \( n \leq 60 \), \( m_1 \leq 6, m_2 \leq 3 \). It is a matter of straightforward computation to check that there are the following 6 solutions of \( G_n = \pm 3^m 7^{m_2} \): \( G_1 = 3, G_2 = -1, G_3 = -7, G_5 = 9, G_7 = 1, G_{17} = 441 \).

9. A Mixed Quadratic-Exponential Equation. In this section, we give an application of the preceding algorithm to the following diophantine equation. Let

\[
\Phi(X, Y) = aX^2 + bXY + cY^2
\]

be a quadratic form with integral coefficients, such that \( D = b^2 - 4ac < 0 \). Let \( q, v, w \) be nonzero integers, and \( p_1, \ldots, p_t \) prime numbers. Consider the equation

\[
\Phi(X, Y) = vq^n
\]

in integers \( X, n \geq 0, m_i \geq 0 (i = 1, \ldots, t) \).

Let \( \beta, \bar{\beta} \) be the roots of \( \Phi(x, 1) \). Let \( h \) be the class number of \( \mathcal{O}(\sqrt{D}) \). There exists a \( \pi \in \mathcal{O}(\sqrt{D}) \) such that we have the principal ideal equation \( (\pi)(\bar{\pi}) = (q^h) \). Put \( n = n_1 + hn_2 \), with \( 0 \leq n_1 < h \). Then \( \Phi(X, Y) = vq^n \) is equivalent to finitely many ideal equations

\[
(aX - a\beta Y)(aX - a\bar{\beta} Y) = (\sigma)(\bar{\sigma})(\pi)^{n_1} (\bar{\pi})^{n_2},
\]
with \((\sigma)(\bar{\sigma}) = (avq^n)\). Hence we have the equations (in algebraic numbers)

\[
\begin{align*}
& aX - a\beta Y = \gamma n2, \\
& aX - a\bar{\beta} Y = \bar{\gamma} n2,
\end{align*}
\]

where \(\gamma\) is composed of units, common divisors of \(aX - a\beta Y\), \(aX - a\bar{\beta} Y\), and \(\sigma\). Notice that there are only finitely many choices for \(\gamma\) possible. Thus, (9.1) is equivalent to a finite number of equations

\[a(\bar{\beta} - \beta)wp_1 \cdots \bar{p}_t = \gamma n2 - \bar{\gamma} n2,\]

or, if we put \(\lambda = \gamma/a(\bar{\beta} - \beta)\) and \(G_{n2} = \lambda n2 + \bar{\lambda} n2\),

\[G_{n2} = wp_1 \cdots p_t.\]

Here \(\{G_{n2}\}_{n2=0}^{\infty}\) is a recurrence sequence with negative discriminant. So (9.2) is of type (1.1), and it can thus be solved by the method presented in Sections 7 and 8.

Before giving an example, we remark that Eq. (9.1) with \(D > 0\) is not solvable with our method. This is due to the fact that in \(Q(\sqrt{D})\) with \(D > 0\) there are infinitely many units, hence infinitely many possibilities for \(\gamma\). Another generalization of Eq. (9.1) is to replace \(q^n\) by \(q_1^n \cdots q_s^n\). This problem is also not solvable by our method, since it does not lead to a binary recurrence sequence if \(s > 2\). It seems that these problems can be solved by using multi-dimensional approximation techniques. This is the subject of further investigations by the author.

We finally present an example.

**Theorem 9.1.** The equation

\[X^2 - 3m_1 m_2 X + 2(3m_1 m_2)^2 = 11 \cdot 2^n\]

in integers \(X, n \geq 0, m_1 \geq 0, m_2 \geq 0\) has only the following solutions:

\[
\begin{array}{cccccccc}
 n & m_1 & m_2 & X & n & m_1 & m_2 & X \\
 1 & 1 & 0 & -1, & 4 & 5 & 2 & 0 & -10, & 19 \\
 1 & 0 & 0 & -4, & 5 & 6 & 0 & 0 & -26, & 27 \\
 2 & 0 & 0 & -6, & 7 & 7 & 0 & 0 & -37, & 38 \\
 3 & 0 & 1 & 2, & 5 & 7 & 3 & 0 & 2, & 25 \\
 3 & 1 & 0 & -7, & 10 & 11 & 1 & 1 & -137, & 158 \\
 4 & 0 & 1 & -6, & 13 & 17 & 2 & 2 & -829, & 1270
\end{array}
\]

**Sketch of Proof.** Put \(\beta = (1 + \sqrt{-7})/2\). Then

\[X^2 - XY + 2Y^2 = (X - \beta Y)(X - \bar{\beta} Y).\]

Notice that \(Q(\sqrt{-7})\) has class number 1, and that

\[2 = (1 + \sqrt{-7})/2 \times (1 - \sqrt{-7})/2, \quad 11 = (2 + \sqrt{-7})(2 - \sqrt{-7}).\]

Suppose \(\gamma | X - \beta Y\) and \(\gamma | X - \bar{\beta} Y\). Then \(\gamma | (\bar{\beta} - \beta)Y = -\sqrt{-7} 3m_1 m_2\). On the other hand, \(\gamma | 11 \cdot 2^n\). It follows that \(\gamma = \pm 1\); hence \(X - \beta Y\) and \(X - \bar{\beta} Y\) are coprime. Thus we have two possibilities:

\[
\begin{align*}
X - \beta Y &= \pm (2 \pm \sqrt{-7})\left(\frac{1 + \sqrt{-7}}{2}\right)^n, \\
X - \bar{\beta} Y &= \pm (2 \mp \sqrt{-7})\left(\frac{1 + \sqrt{-7}}{2}\right)^n,
\end{align*}
\]
in each equation the 2nd and 3rd ± being independent. Hence, we have to solve
\[(9.3) \quad G^{(j)}_n = \lambda^{(j)} B^n + \bar{\lambda}^{(j)} \bar{B}^n = 3^{m_1} 7^{m_2} \quad (j = 1, 2),\]
with \[G^{(j)}_{n+1} = G^{(j)}_n - 2G^{(j)}_{n-1} \quad (j = 1, 2)\]
and \[\lambda^{(1)} = \bar{\lambda}^{(2)} = (2 + \sqrt{-7})/\sqrt{-7},\]
so that \[G^{(1)}_0 = G^{(2)}_0 = 1, \ G^{(1)}_1 = 3, \ G^{(2)}_1 = -1.\]
Notice that \[\theta_i^{(1)} = -\theta_i^{(2)} \quad (i = 1, 2),\]
and \[\psi^{(1)} = -\psi^{(2)}.\]
For \(j = 1\) we solved (9.3) in the example of Subsection 8C. We leave it to the reader to solve (9.3) for \(j = 2\); this can be done with the numerical data given in Subsection 8C. □

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